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CM-triviality and relational structures

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Abstract

Continuing work of Baldwin and Shi (Ann. Pure Appl. Logic 79 (1996) 1), we study non-ω-saturated generic structures of the ab initio Hrushovski construction with amalgamation over closed sets. We show that they are CM-trivial with weak elimination of imaginaries. Our main tool is a new characterization of non-forking in these theories.

MSC: 03C45

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1. Introduction

By amalgamating a suitable class of relational finite structures, Hrushovski [11] constructed new strongly minimal sets which are not one-based and do not interpret groups either. He also introduced the notion of CM-triviality, which is a weaker property of the geometry of forking than one-basedness. Finally, he showed that his new strongly minimal sets (more generally, flat structures of finite Morley rank) are CM-trivial with weak elimination of imaginaries. In these constructions, the number of points and the number of tuples satisfying relations satisfy a linear inequality; we call them ab initio, and the resulting structures *generic*.

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This was generalized by Baudisch [5], who constructed a new $\aleph_1$-categorical CM-trivial group with weak elimination of imaginaries by replacing the number of tuples satisfying a relation in the ab initio construction by the dimension of an appropriate vector space. The ab initio construction yields important examples as in [1,9,10,13]. All the generic structures mentioned above are ω-saturated. On the other hand, non-ω-saturated generic structures in a finite relational language have been studied by Baldwin, Shelah and Shi, who show that they are stable [4] and near model-complete [2,12]. There are many papers related to generic structures, for example [4,17,3]; for further results see [16].

CM-triviality is also studied in many papers, for example [6–8,14,15,18]. In the introduction of [6], the authors point out that the ab initio constructions produce theories which are CM-trivial, or at least CM-trivial *over* the data. (The latter notion has not been made precise as yet.) Following a suggestion by Pillay, the second author showed in [19] that the theories of ω-saturated generic structures are stable CM-trivial with weak elimination of imaginaries; this holds in particular for Herwig's theory in [9]. The main tools in [19] are a characterization of non-forking and stationarity of types over algebraically closed sets in the real sort.

In this paper, we show that the same results hold without assuming ω-saturation. For any structure, we will consider the notion of closure, called *intrinsic closure*, defined by a predimension δ (local rank) on its substructures. We say that a theory has *finite closure* if the size of the closure of any finite substructure of any of its models is finite. The theory of an ω-saturated generic structure has finite closure, but this need not hold without ω-saturation; for instance the generic structures studied in [4] do not have finite closure.

In the finite-closure case, the relation between the local rank δ and the global rank d is relatively straightforward (see [17]). This relation is more complicated in the infinite-closure case; if the language is finite, Baldwin and Shi [4] have analyzed the difference between the two cases, using approximations of $d(B/A)$ in terms of $d(B/A')$ for various finite A', which was sufficient for their main results.

While Baldwin and Shi considered the theory of a generic structure, we will consider completions of the universal theory T_0, where $M \models T_0$ iff $\delta(A) \geqslant 0$ for any finite substructure A of M, satisfying in addition a certain condition called *amalgamation over closed sets*. For an arbitrary (possibly infinite) language, we shall characterize non-forking in any L-completion T of T_0 with amalgamation over closed sets (even for infinite closure), and show stationarity of types over algebraically closed sets in the real sort. From this we shall deduce stability, weak elimination of imaginaries, and CM-triviality.

In the next section, we quickly review definitions and facts of [4], describe the notions of independence, and summarize the known facts. In the third section, we shall develop a calculus of relative predimension $\delta(A/B)$ even for infinite B, and show that the dimension $d(A/B)$ equals the predimension of the respective closures $\delta(\overline{AB}/\bar{B})$. This will be applied to prove equivalence of the three notions of independence: non-forking, dimension-independence, and closed free amalgamation, as well as stationarity of types over algebraically closed sets in the real sort. In the fourth section we discuss weak elimination of imaginaries and CM-triviality.

In an appendix we give a correct proof of 4.8 of [4], which contained many typos.

Historical remarks: In the case of a finite language, Baldwin and Shi claim to show equivalence of non-forking and dimension-independence for algebraically closed sets A, B (over $A \cap B$) in any completion of T_0 with free amalgamation over closed sets [4, Lemma 3.38], as well as weak elimination of imaginaries [4, Lemmas 5.5 and 5.6]. However, they assume that dimension-independence coincides with closed free amalgamation; the second author noticed that their proof that dimension-independence implies closed free amalgamation [4, Lemmas 3.31 and 3.33] needed [4, Lemma 3.26], which is incorrect, as they erroneously assume that $B = A \cap C$ implies $\bar{B} = \bar{A} \cap \bar{C}$, where $\bar{X}$ denotes intrinsic closure of X.

In an attempt to repair the errors in [4], the second author showed equivalence of non-forking and closed free amalgamation, as well as weak elimination of imaginaries and CM-triviality in an earlier version of this paper. He also managed to show that dimension-independence implies closed free amalgamation; however, the converse remained open and was solved by the first author by introducing the notion of a set A *calculable* over a set B, and a way to compute $\delta(A/B)$. Thus, the development from Definition 3.1 to Corollary 3.18 is due to Verbovskiy and the rest is due to Yoneda. Both authors would like to thank Baldwin for many helpful suggestions and discussions, and Wagner for the generalization of Sections 2–4 from finite languages to infinite languages. They would also like to express many thanks to Wagner and the editor for very kind advice on improving this paper.

2. Preliminaries

Notation. Our notation is standard. For simplicity we write AB for $A \cup B$. For a tuple $\bar{a}$ we write $\bar{a} \in A$ if every member of the tuple $\bar{a}$ is an element of A. And we write $A \subseteq_\omega B$ if A is a finite subset of B.

Let $L = \{R_i(\bar{x}) : i \in I\}$ be a relational language, where the R_i are relation symbols of arity n_i. In this paper, we only consider L-structures where the R_i are closed under permutation, and $R_i(a_1, \ldots, a_{n_i})$ implies $a_j \neq a_k$ for $1 \leqslant j < k \leqslant n_i$.

Fix $\alpha_i \in \mathbb{R}^+$ for $i \in I$. For a finite L-structure A we define a predimension δ as follows:

- $\delta(A) = |A| - \sum_{i \in I} \alpha_i \cdot |R_i^A|$, where A is a finite L-structure, and R_i^A is the set of sequences (up to permutation) of A satisfying R_i.
- If A and B are finite L-structures, we put $\delta(A/B) = \delta(AB) - \delta(B)$.

We put $r_i(A,B) = \{\bar{d} \in R_i^{AB} : \bar{d} \cap A \neq \emptyset$ and $\bar{d} \cap B \neq \emptyset\}$, and $e(A,B) = \sum_{i \in I} \alpha_i \cdot |r_i(A,B)|$. Note that if $A \cap B = \emptyset$, then $\delta(A/B) = \delta(A) - e(A,B)$.

Let A, B, C be disjoint sets. We put $r_i(A,B,C) = \{\bar{d} \in R_i^{ABC} : \bar{d} \cap A \neq \emptyset$ and $\bar{d} \cap C \neq \emptyset\}$, and $e(A,B,C) = \sum_{i \in I} \alpha_i \cdot |r_i(A,B,C)|$.

In this paper, for $\alpha = (\alpha_i : i \in I)$ let $\mathbf{K}_\alpha$ be the class of finite L-structures whose substructures all satisfy $\delta(.) \geqslant 0$, and let $\bar{\mathbf{K}}_\alpha$ be the class of all L-structures whose finite substructures are all in $\mathbf{K}_\alpha$. Note that $\bar{\mathbf{K}}_\alpha$ is universally axiomatized by the set T_0 of sentences $\forall \bar{x} \neg \mathrm{Diag}_A^{I_0}(\bar{x})$, where $I_0 \subseteq_\omega I$ and A is a finite $\{R_i : i \in I_0\}$-structure with $\delta(A) < 0$ and diagram $\mathrm{Diag}_A^{I_0}$.

We briefly recall the notions of free join, closedness, generic structures, amalgamation over closed sets and intrinsic closure.

Free join: Let A, B, C be possibly infinite L-structures. We say that A and B are *freely joined* over C, written $ABC = A \otimes_C B$, if $A \cap B \subseteq C$ and $r_i(A - C, C, B - C) = 0$ for all $i \in I$. In other words, A and B are freely joined over C if the relations on ABC are those on AC together with those on BC.

Closedness: For finite L-structures $A \subset B$, we say that B is a *minimal intrinsic extension* of A if $\delta(B/A) < 0$ and $\delta(A'/A) \geqslant 0$ for any $A' \subset B$. We write $A \subset_{\min} B$. For possibly infinite L-structures $A \subseteq B$, we say that A is *closed* in B and write $A \leqslant B$ if whenever $A_0 \subset_{\min} B_0 \subset_\omega B$ and $A_0 \subset A$, then $B_0 \subset A$. A map f is a *closed embedding* of A into B if f is an embedding as L-structures and $f(A) \leqslant B$. Let M be an infinite L-structure. We say that M has *finite closure* if for any finite substructure A of M there exists a finite L-structure B such that $A \subseteq B \leqslant M$. Finally, a complete L-theory T has *finite closure* if any model of T does.

Generic structures: Let $\mathbf{K}$ be a subclass of $\mathbf{K}_\alpha$ closed under isomorphisms and substructures, and $\bar{\mathbf{K}}$ the class of L-structures whose finite substructures are in $\mathbf{K}$. We say that an L-structure M is $(\mathbf{K}, \leqslant)$-*generic* if M is countable and has the following two properties: (1) Any finite L-substructure of M is in $\mathbf{K}$. (2) If $A, B \in \mathbf{K}$ are such that $A \leqslant M$ and $A \leqslant B$, then M contains a copy of B over A, say B', such that $B' \leqslant M$. Note that any two $(\mathbf{K}, \leqslant)$-generic structures with finite closure are elementarily equivalent. It is well known that if α is irrational, then a $(\mathbf{K}, \leqslant)$-generic structure is not ω-saturated and does not have finite closure.

Amalgamation over closed sets: An L-completion T of T_0 has *amalgamation over closed sets* if for all models N_1, N_2 of T, any model A of $T_\forall$ and closed embeddings f_i of A into N_i for $i = 1, 2$, there exist a model N of T and elementary embeddings g_i of N_i into N such that $g_1 \circ f_1 = g_2 \circ f_2$. We say that T has *free amalgamation over closed sets* if $g_1(N_1)g_2(N_2) = g_1(N_1) \otimes_{g_i(f_i(A))} g_2(N_2)$, and *closed free amalgamation over closed sets* if $g_1(N_1)g_2(N_2) = g_1(N_1) \otimes_{g_i(f_i(A))} g_2(N_2) \leqslant N$. (While [4] defined these notions for the theories of generic structures, we define them for L-completions of T_0.)

Intrinsic closure: Let N be an infinite L-structure and A a finite L-substructure of N. We define the following sets inductively: $\mathrm{icl}_N^{-1}(A) = A$, and

$$\mathrm{icl}_N^{i+1}(A) = \bigcup \{B : \exists A' \subset_\omega \mathrm{icl}_N^i(A) : A' \subset_{\min} B \subset_\omega N\} \cup \mathrm{icl}_N^i(A).$$

The *intrinsic closure* of A in N is defined as $\mathrm{icl}_N(A) = \bigcup_{i<\omega} \mathrm{icl}_N^i(A)$. For an infinite L-substructure A of N, we put $\mathrm{icl}_N(A) = \bigcup \{\mathrm{icl}_N(A_0) : A_0 \subset_\omega A\}$; we shall write $\bar{A}$ for $\mathrm{icl}_N(A)$ if its meaning is clear from the context. By definition we see that $\mathrm{icl}_N(A) \leqslant N$ for any (possibly infinite) L-structure $A \subset N$, and if $A \subseteq C \leqslant N$, then $\mathrm{icl}_N(A) \subseteq C$ (i.e. minimality). It is easy to see that if $A \subset N \in \overline{\mathbf{K}_\alpha}$, then $\mathrm{icl}_N(A) \subseteq \mathrm{acl}_N(A)$; it follows that if $A \leqslant N$ and $N \prec M$, then $A \leqslant M$.

Definition 2.1. Let $A, B \subseteq_w M$. Put $d_M(A) = \inf\{\delta(A') : A \subseteq A' \subseteq_w M\}$ and $d_M(A/B) = d_M(AB) - d_M(B)$. For infinite B we put $d_M(A/B) = \inf\{d_M(A/B_0) : B_0 \subseteq_\omega B\}$.

We shall omit the index M if it is clear from the context and write $d(A)$ and $d(A/B)$. Clearly, if $A \subset_\omega M$ and $B \subseteq C \subseteq M$ are infinite, then $d_M(A/B) \geqslant d_M(A/C)$. We shall see later (Lemma 3.12) that for finite B the two definitions agree.

Definition 2.2. (1) Let $A \subseteq_\omega M$ and $B, C \subseteq M$. We say that A is *d-independent* from B over C, denoted by $A \mathop{\smile\!\!\!\!\!\!|}^{d}_{C} B$, if $d(A/BC) = d(A/C)$ and $\overline{AC} \cap \overline{BC} = \bar{C}$.

(2) Let $A, B, C \subseteq M$, and A infinite. We say that A is *d-independent* from B over C, denoted by $A \mathop{\smile\!\!\!\!\!\!|}^{d}_{C} B$, if A_0 is d-independent from B_0 over C for each finite $A_0 \subseteq_\omega A$ and $B_0 \subseteq_\omega B$.

We shall see later (Remark 3.17) that the two definitions agree for finite A.

We now introduce another independence relation, Φ-independence.

Definition 2.3. We say that subsets $A, B \subseteq N$ are Φ_N-independent over C, denoted $A \mathop{\smile\!\!\!\!\!\!|}^{\Phi}_{C} B$, if $\mathrm{acl}(AC) \cap \mathrm{acl}(BC) = \mathrm{acl}(C)$ and $\mathrm{acl}(AC) \cup \mathrm{acl}(BC) = \mathrm{acl}(AC) \otimes_{\mathrm{acl}(C)} \mathrm{acl}(BC) \leqslant N$. We shall usually suppress the subscript N if it is clear from the context.

Let T be any L-completion of T_0, and $\mathscr{M}$ a big model of T. We thus have three notions of independence for algebraically closed sets A and B (over $A \cap B$):

(1) $A \mathop{\smile\!\!\!\!\!\!|}_{A \cap B} B$ (non-forking);
(2) $A \mathop{\smile\!\!\!\!\!\!|}^{d}_{A \cap B} B$ (d-independence);
(3) $A \mathop{\smile\!\!\!\!\!\!|}^{\Phi}_{A \cap B} B$ ($\Phi_{\mathscr{M}}$-independence).

Fact 2.4. (1) [17,19] *If the complete L-theory* $\mathrm{Th}(M)$ *of an ω-saturated generic structure M has finite support, then the above three notions on independence are equivalent in* $\mathrm{Th}(M)$. (*We say that* $\mathrm{Th}(M)$ *has finite support if for any finite model A of* $\mathrm{Th}(M)_\forall$ *the set* $\{i \in I : R_i^A \neq \emptyset\}$ *is finite.*)

(2) *Suppose L is finite, let T be an L-completion of* T_0, *and* $\mathscr{M}$ *a big model of T.*

(a) [4, Lemma 3.35] *If T has amalgamation over closed sets, then non-forking coincides with d-independence over* $\aleph_1$*-saturated models of T.*

(b) [3, Fact 2.3] *If T is stable, then* $A \mathop{\smile\!\!\!\!\!\!|}_{A \cap B} B$ *implies* $AB = A \otimes_{A \cap B} B$ *for any algebraically closed sets* $A, B \subset \mathscr{M}$.

3. The three notions of independence are equivalent in any *L*-completion *T* of T_0 with amalgamation over closed sets

In the first half of this section, we show that d-independence coincides with Φ-independence in T_0; in the second half, we prove that non-forking coincides with Φ-independence in any L-completion T of T_0 with amalgamation over closed sets. Thus, all three notions of independence are equivalent in such a theory.

In this section let M be a model of T_0. We shall work in M and write $d(A)$ for $d_M(A)$ and $\bar{A}$ for $\mathrm{icl}_M(A)$. We shall assign a reasonable meaning to $\delta(B/A)$ for possibly infinite $A, B \subseteq M$ satisfying a certain condition, and prove that $d(B/A) = \delta(\overline{AB}/\bar{A})$ for any finite B. Consequently, we obtain that $A \mathop{\smile\!\!\!\!\!\!\,|}^{d}_{C} B$ iff $\overline{AC} \cup \overline{BC} = \overline{AC} \otimes_{\bar{C}} \overline{BC} \leqslant M$ for any $A, B, C \subseteq M$.

Definition 3.1. A subset A of M is *almost closed* if there is a positive real γ such that $e(B, A) \leqslant \delta(B) + \gamma$ for every finite B disjoint from A. The least such γ will be denoted t_A.

Clearly, any finite set is almost closed.

Definition 3.2. Let $A \subseteq M$ be possibly infinite, and $B \subseteq_\omega M$. We extend the definition of $\delta(B/A)$, and of a minimal intrinsic extension, as follows:

- $\delta(B/A) = \delta(B - A) - e(B - A, A)$.
- $A \subset_{\min} AB$ if $-\infty < \delta(B/A) < 0$ and $\delta(B'A) \geqslant 0$ for any $B' \subset B$. In this case, we say that B is a *minimal intrinsic extension* of A. Sometimes we write $A \subset_{\min} B$ even if $A \not\subseteq B$, meaning $A \subset_{\min} AB$.

If A and B are finite, this definition agrees with the old one. If A is infinite, $\delta(B/A)$ may equal $-\infty$, but if A is almost closed, then $\delta(B/A) \geqslant -t_A > -\infty$. Clearly, if $A' \subseteq A$ with $B \cap A = B \cap A'$, then $\delta(B/A) \geqslant \delta(B/A')$ (monotonicity).

Remark 3.3. Note that for finite B,

$$e(B, A) = \lim_{A_0 \subseteq_\omega A} e(B, A_0),$$

where the limit is with respect to the directed set of finite subsets of A. Hence for disjoint A, B, C with BC finite,

$$\begin{aligned} e(BC, A) &= \lim_{A_0 \subseteq_\omega A} e(BC, A_0) = \lim_{A_0 \subseteq_\omega A} [e(B, A_0C) + e(C, A_0) - e(B, C)] \\ &= \lim_{A_0 \subseteq_\omega A} e(B, A_0C) + \lim_{A_0 \subseteq_\omega A} e(C, A_0) - e(B, C) \\ &= e(B, AC) + e(C, A) - e(B, C), \end{aligned}$$

in particular, the limit on the first line exists iff the two limits on the second line do. It follows that for disjoint A, B, C with BC finite, if $\delta(BC/A) > -\infty$ then

$$\begin{aligned} \delta(BC/A) &= \delta(BC) - e(BC, A) \\ &= \delta(B) + \delta(C) - e(B, C) - [e(B, AC) + e(C, A) - e(B, C)] \\ &= \delta(B) - e(B, AC) + \delta(C) - e(C, A) = \delta(B/AC) + \delta(C/A). \end{aligned}$$

Lemma 3.4. *Suppose that $A \leqslant M$ and $C \subset_\omega M$, with $A \cap C = \emptyset$. Then $e(C, A) \leqslant \delta(C)$ and AC is almost closed in M.*

Proof. For any $A_0 \subset_\omega A$ we have that $0 \leqslant \delta(C/A_0) = \delta(C) - e(C, A_0)$, so

$$e(C, A) = \lim_{A_0 \subseteq_\omega A} e(C, A_0) \leqslant \delta(C).$$

To show that AC is almost closed in M, consider a finite B disjoint from AC. Then $e(B,C)=\delta(B)+\delta(C)-\delta(BC)$, and $A\leqslant M$ implies $e(BC,A)\leqslant\delta(BC)$. Since $e(C,A)\geqslant 0$,

$$\begin{aligned} e(B,AC) &= e(BC,A)+e(B,C)-e(C,A) \\ &\leqslant \delta(BC)+\delta(B)+\delta(C)-\delta(BC) \leqslant \delta(B)+\delta(C). \qquad \square \end{aligned}$$

Lemma 3.5. *Let $(B_n)_{n<\omega}$ be an increasing chain of minimal intrinsic extensions, i.e. $B_n\subset B_{n+1}$ and $B_n\subset_{\min} B_{n+1}$. Let $F\subseteq_\omega \bigcup_{n<\omega} B_n$. Then there exists n such that $\delta(F/B_0)\geqslant\delta(B_n/B_0)$.*

Proof. We show by induction on n that $\delta(F\cap B_n/B_0)\geqslant\delta(B_n/B_0)$. As there is $n<\omega$ with $F\subseteq B_n$, this will show the lemma. For $n=0$ the assertion is trivial. Suppose that the induction hypothesis holds for $n<\omega$. If $\delta(B_{n+1}/B_0)=-\infty$ we are done. Otherwise,

$$\begin{aligned} \delta(B_{n+1}/B_0) &= \delta(B_{n+1}/B_n)+\delta(B_n/B_0) \leqslant \delta(B_{n+1}\cap F/B_n)+\delta(B_n\cap F/B_0) \\ &\leqslant \delta(B_{n+1}\cap F/B_0(B_n\cap F))+\delta(B_n\cap F/B_0)=\delta(B_{n+1}\cap F/B_0), \end{aligned}$$

where the first inequality holds because $B_n\subset_{\min} B_{n+1}$ and by inductive hypothesis, and the second one holds by monotonicity. $\square$

Lemma 3.6. *Let $A\subseteq M$ be almost closed, possibly infinite. Let $(B_n)_{n<\omega}$ be an increasing chain of minimal intrinsic extensions such that $A\subseteq B_0$, B_0-A is finite, and $B_n\subset_{\min} B_{n+1}$. Then*

(1) $\lim_{n\to\omega}\delta(B_n/A)$ *exists and is finite*;
(2) $B:=\bigcup_{n<\omega} B_n$ *is almost closed*;
(3) $\lim_{n\to\omega}\delta(B_n/A)=\delta(B_k/A)+\lim_{n\to\omega}\delta(B_n/B_k)=\delta(B_k/A)+\sum_{n=k}^{\omega}\delta(B_{n+1}/B_n)$ *for any $k<\omega$; in particular*, $\lim_{n\to\omega}\delta(B_{n+1}/B_n)=0$; *and*
(4) *if $(B'_n)_{n<\omega}$ is another such chain with $\bigcup_{n<\omega} B'_n=B$, then* $\lim_{n\to\omega}\delta(B_n/A)=\lim_{n\to\omega}\delta(B'_n/A)$.

Proof.
(1) Since $B_n\subset_{\min} B_{n+1}$, we get $\delta(B_{n+1}/B_n)<0$, whence for all $n<\omega$

$$-t_A \leqslant \delta(B_{n+1}/A)=\delta(B_{n+1}/B_n)+\delta(B_n/A) < \delta(B_n/A).$$

Hence the sequence $(\delta(B_n/A))_{n<\omega}$ is decreasing and bounded, and its limit exists.
(2) Consider a finite C disjoint of B. As A is almost closed, $\delta(B_nC/A)>-\infty$, and

$$\delta(C/B_n)+\delta(B_n/A)=\delta(B_nC/A)\geqslant -t_A.$$

Moreover, $\delta(C/B_{n+1})\leqslant\delta(C/B_n)$ by monotonicity, so $\lim_{n\to\omega}\delta(C/B_n)$ exists, and

$$\begin{aligned} \delta(C/B) &= \delta(C)-e(C,B)=\delta(C)-\lim_{n\to\omega} e(C,B_n) \\ &= \lim_{n\to\omega}\delta(C/B_n)\geqslant -t_A-\lim_{n\to\omega}\delta(B_n/A). \end{aligned}$$

Thus B is almost closed.

(3) Obvious.
(4) By the above, $(\delta(B'_n/A))_{n<\omega}$ is a decreasing bounded sequence. By Lemma 3.5 for any $k<\omega$ there is $n<\omega$ with $\delta(B_k/A) \geqslant \delta(B'_n/A)$, whence $\lim_{k\to\omega} \delta(B_k/A) \geqslant \lim_{n\to\omega} \delta(B'_n/A)$. The reverse inequality follows from symmetry. □

Definition 3.7. A set B is *calculable* over a set A if A is almost closed and there is an increasing chain $(B_n)_{n<\omega}$ with $A \subseteq B_0$, $B_0 - A$ finite, $B_n \subset_{\min} B_{n+1}$ and $AB = \bigcup_{n<\omega} B_n$. If B is calculable over A, we define $\delta(B/A) = \lim_{n\to\omega} \delta(B_n/A)$.

Lemma 3.6 provides that $\delta(B/A)$ is well defined. Note that B is calculable over A iff $B \cup A$ is iff $B - A$ is.

Our aim is to show that if A is closed and B is finite, then $\delta(\overline{BA}/A)$ is defined and equal to $d(B/A)$, as in the finite-closure case.

Lemma 3.8. *Let A be almost closed. If B is calculable over A and C is calculable over AB, then BC is calculable over A and $\delta(BC/A) = \delta(C/AB) + \delta(B/A)$. In particular, if B is calculable over A and A is calculable over $\emptyset$, then $\delta(B/A) = \delta(BA) - \delta(A)$.*

Proof. Let $(B_n)_{n<\omega}$ and $(C_n)_{n<\omega}$ be corresponding increasing chains of minimal intrinsic extensions, with $A \subseteq B_0$, $AB \subseteq C_0$, and $B_0 - A$ and $C_0 - AB$ finite. Put $C'_n = C_n - AB$, for $n<\omega$. As

$$
\begin{aligned}
0 &> \delta(C_{n+1}/C_n) = \delta(C_{n+1} - C_n) - e(C_{n+1} - C_n, C_n) \\
&= \delta(C'_{n+1} - C'_n) - \lim_{k\to\omega} e(C'_{n+1} - C'_n, B_k C'_n) = \lim_{k\to\omega} \delta(C'_{n+1}/B_k C'_n),
\end{aligned}
$$

there is $\tau(n)<\omega$ such that $\delta(C'_{n+1}/B_{\tau(n)}C'_n)<0$; clearly we may assume that τ is an increasing function.

Claim. *We may choose τ such that for all $n<\omega$*

$$
0 \leqslant \delta(C'_n/B_{\tau(n)}) - \delta(C'_n/AB) < \frac{1}{n}.
$$

Proof. The first inequality follows from monotonicity, for any τ. For the second, note that

$$
\delta(C'_n/B_k) - \delta(C'_n/AB) = e(C'_n, AB) - e(C'_n, B_k)
$$

(which is finite by almost closedness of B); as $\lim_{k\to\omega} e(C'_n, B_k) = e(C'_n, AB)$, we can choose $\tau(n) = k$ sufficiently large to satisfy the second inequality. □

Consider the increasing sequence

$$
B_{\tau(0)}C'_0, B_{\tau(0)}C'_1, B_{\tau(0)+1}C'_1, B_{\tau(0)+2}C'_1, \ldots, B_{\tau(1)}C'_1, B_{\tau(1)}C'_2, B_{\tau(1)+1}C'_2, \ldots\ .
$$

Clearly $B_{\tau(0)}C'_0 - A$ is finite, as is the difference between two successive sets in the sequence.

Claim. *This sequence can be refined to an increasing chain of minimal intrinsic extensions.*

Proof. By monotonocity of δ and as $B_n \subset_{\min} B_{n+1}$, for any $k, n < \omega$ and $B_n \subseteq B' \subset B_{n+1}$

$$\delta(B_{n+1}/B'C'_k) \leqslant \delta(B_{n+1}/B') < 0.$$

On the other hand, for all $n < \omega$ and $C'_n \subseteq C' \subset C'_{n+1}$

$$\delta(C'/B_{\tau(n)}C'_n) \geqslant \delta(C'/ABC'_n) \geqslant 0$$

by monotonicity of δ and since $C_n \subset_{\min} C_{n+1}$. Hence

$$\delta(C'_{n+1}/B_{\tau(n)}C') = \delta(C'_{n+1}/B_{\tau(n)}C'_n) - \delta(C'/B_{\tau(n)}C'_n) < 0$$

by choice of τ (note that all sets in the sequence are almost closed). This proves the claim. □

Therefore, BC is calculable over A. But now

$$\begin{aligned}\delta(BC/A) &= \lim_{n\to\omega} \delta(B_{\tau(n)}C'_n/A) = \lim_{n\to\omega} [\delta(C'_n/B_{\tau(n)}) + \delta(B_{\tau(n)}/A)]\\ &= \lim_{n\to\omega} \delta(C'_n/B_{\tau(n)}) + \lim_{n\to\omega} \delta(B_{\tau(n)}/A)\\ &= \lim_{n\to\omega} \delta(C'_n/AB) + \delta(B/A) = \delta(C/AB) + \delta(B/A). \qquad \square\end{aligned}$$

Lemma 3.9. *Let $A \subseteq M$ be possibly infinite and almost closed. Then $\bar{A}$ is calculable over A. In particular $|\bar{A} - A| \leqslant \aleph_0$.*

Proof. First note that if B is almost closed, C, D are finite, $B \subset_{\min} C$ and $C \nsubseteq D$, then

$$\delta(C/BD) \leqslant \delta(C/B(D \cap C)) = \delta(C/B) - \delta(D \cap C/B) \leqslant \delta(C/B).$$

Next, suppose that there is a sequence $(C_i)_{i<\omega}$ of minimal intrinsic extensions of B with $\delta(C_i/B) \leqslant -1/n$ and $C_i \nsubseteq B \bigcup_{j<i} C_j$. Choose $k > nt_B$. Then

$$\delta\left(\bigcup_{i<k} C_i/B\right) = \sum_{i<k} \delta\left(C_i/B \bigcup_{j<i} C_j\right) \leqslant \sum_{i<k} \delta(C_i/B) \leqslant -k/n < -t_B,$$

a contradiction. Hence, there are only finitely many minimal intrinsic extensions C with $\delta(C/B) \leqslant -1/n$.

It follows that we can construct a sequence $A = A_0 \subset_{\min} A_1 \subset_{\min} A_2 \subset_{\min} \cdots$ such that $\delta(A_{n+1}/A_n)$ is minimal possible given A_n, for all $n < \omega$. (If at some stage A_n is closed, we put $A_{n+1} = A_n$.) Put $A' = \bigcup_{n<\omega} A_n$. Then A' is calculable over A, and $\lim_{n\to\omega} \delta(A_{n+1}/A_n) = 0$.

Clearly $A' \subseteq \bar{A}$; suppose that A' is not closed. Hence there is C disjoint from A' with $A' \subset_{\min} A'C$; since

$$0 > \delta(C/A') = \delta(C) - e(C,A') = \delta(C) - \lim_{n\to\omega} e(C,A_n) = \lim_{n\to\omega} \delta(C/A_n),$$

there is $n<\omega$ with $\delta(C/A_n)<0$; it follows from monotonicity that $A_k \subset_{\min} A_kC$ for all $k \geqslant n$. As $\delta(A_{k+1}/A_k) \leqslant \delta(C/A_k) \leqslant \delta(C/A_n)$ by our choice of the sequence $(A_i)_{i<\omega}$, this contradicts $\lim_{k\to\omega} \delta(A_{k+1}/A_k)=0$. □

Lemma 3.10. *Let $B \subseteq_\omega M$. Then $d(B)=\delta(\bar{B})$.*

Proof. Let $(B_n)_{n<\omega}$ be a chain of minimal intrinsic extensions with $B_0=B$ and $\bigcup_{n<\omega} B_n=\bar{B}$. Then $\delta(B_n)\geqslant d(B)$ for each $n<\omega$ by definition of d, so $\delta(\bar{B})\geqslant d(B)$.

Take an arbitrary finite F with $B\subseteq F$. Since $\bar{B}$ is closed, $\delta(F\cap\bar{B})\leqslant\delta(F)$. By Lemma 3.5 there exists n such that $\delta(B_n)\leqslant\delta(F\cap\bar{B})$. Thus $\delta(\bar{B})\leqslant\delta(B_n)\leqslant d(B)$, as desired. □

Remark 3.11. By Lemmas 3.8 and 3.10 we get for $A,B \subset_\omega M$:

$$d(A/B) = d(AB) - d(B) = \delta(\overline{AB}) - \delta(\bar{B}) = \delta(\overline{AB}/\bar{B}).$$

Monotonicity of d-rank has been shown in [4] for a finite language. Here we give an independent proof for the general case of an infinite language.

Lemma 3.12. *Let $A \subseteq_\omega M$ and $B \subseteq C \subseteq_\omega M$. Then $d(A/B)\geqslant d(A/C)$.*

Proof. Let $(B_n)_{n<\omega}$ be an increasing chain with union $\overline{AB}$ such that $B_0=A\bar{B}$ and $B_n \subset_{\min} B_{n+1}$; and $(C_n)_{n<\omega}$ an increasing chain with union $\overline{AC}$ such that $C_0=A\bar{C}$ and $C_n \subset_{\min} C_{n+1}$. Then

$$\begin{aligned}
\delta(B_n/\bar{B}) &= \delta(B_n/\bar{B}(\bar{C}\cap B_n)) + \delta(\bar{C}\cap B_n/\bar{B}) \\
&\geqslant \delta(B_n/\bar{B}(\bar{C}\cap B_n)) && \text{since } \bar{B} \text{ is closed} \\
&\geqslant \delta(B_n/\bar{C}) && \text{by monotonicity of } \delta \\
&\geqslant \delta(C_k/\bar{C}) && \text{by Lemma 3.5 for sufficiently large } k<\omega \\
&\geqslant \delta(\overline{AC}/\bar{C}) && \text{by definition and Lemma 3.6} \\
&= d(A/C) && \text{by Remark 3.11.}
\end{aligned}$$

Therefore,

$$\begin{aligned}
d(A/B) &= d(AB) - d(B) = \delta(\overline{AB}) - \delta(\bar{B}) \\
&= \delta(\overline{AB}/\bar{B}) = \lim_{n\to\omega} \delta(B_n/\bar{B}) \geqslant d(A/C). \qquad \square
\end{aligned}$$

Hence $d(A/B)=\inf_{B_0\subseteq B} d(A/B_0)$ for $A,B\subset_\omega M$, and for all $A\subset_\omega M$ and $B\subseteq C\subseteq M$ monotonicity holds: $d(A/B)\geqslant d(A/C)$.

Lemma 3.13. *Let $A\subseteq M$ and $B\subseteq_\omega M$. Then $d(B/A)=\delta(\overline{AB}/\bar{A})$. In particular, $d(B/A) = d(B-\bar{A}/A)=d(B/\bar{A})$.*

Proof. Fix an increasing chain $(B_n)_{n<\omega}$ of minimal intrinsic extensions with $B_0 = \bar{A}B$ and $\bigcup_{n<\omega} B_n = \overline{AB}$. Put $B'_n = B_n - \bar{A}$, a finite set. Since $\delta(B_n/\bar{A}) = \delta(B'_n) - e(B'_n, \bar{A})$, for sufficiently big $A_n \subseteq_\omega A$ we have $e(B'_n, \bar{A}_n) > e(B'_n, \bar{A}) - 1/n$, whence $\delta(B'_n/\bar{A}_n) < \delta(B_n/\bar{A}) + 1/n$. Then

$$\begin{aligned} d(B/A) &\leqslant d(B/A_n) = \delta(\overline{BA_n}) - \delta(\bar{A}_n) \\ &\leqslant \delta(B'_n\bar{A}_n) - \delta(\bar{A}_n) = \delta(B'_n/\bar{A}_n) < \delta(B_n/\bar{A}) + \frac{1}{n}; \end{aligned}$$

whence

$$d(B/A) \leqslant \lim_{n\to\omega} [\delta(B_n/\bar{A}) + \tfrac{1}{n}] = \delta(\overline{AB}/\bar{A}).$$

On the other hand, consider $A' \subseteq_\omega A$, and let $(C_n)_{n<\omega}$ be an increasing chain of minimal intrinsic extensions with $C_0 = \bar{A}'B$ and $\bigcup_{n<\omega} C_n = \overline{A'B}$. Then

$$\begin{aligned} \delta(C_n/\bar{A}') &= \delta(C_n/\bar{A}'(\bar{A}\cap C_n)) + \delta(\bar{A}\cap C_n/\bar{A}') \\ &\geqslant \delta(C_n/\bar{A}'(\bar{A}\cap C_n)) && \text{since } \bar{A}' \text{ is closed} \\ &\geqslant \delta(C_n/\bar{A}) && \text{by monotonicity of } \delta \\ &\geqslant \delta(B_k/\bar{A}) && \text{by Lemma 3.5 for sufficiently large } k<\omega \\ &\geqslant \delta(\overline{AB}/\bar{A}). \end{aligned}$$

Therefore,

$$d(B/A') = \delta(\overline{BA'}/\bar{A}') = \lim_{n\to\omega} \delta(C_n/\bar{A}') \geqslant \delta(\overline{AB}/\bar{A}),$$

and $d(B/A) = \inf_{A'\subseteq_\omega A} d(B/A') \geqslant \delta(\overline{AB}/\bar{A})$.

The last assertion follows from $\overline{BA} = \overline{(B-\bar{A})A}$. □

Theorem 3.14. *Let $A, B \subseteq_\omega M$ and $C \subseteq M$. Then $A \mathop{\smile\!\!\!\!\!\!|}^d_C B$ iff $\overline{AC} \cup \overline{BC} = \overline{AC} \otimes_{\bar{C}} \overline{BC} \leqslant M$.*

Proof. Note that both assertions imply $\overline{AC} \cap \overline{BC} = \bar{C}$. We can therefore assume this equality.

Claim. *$\overline{AC}$ is calculable over $\overline{BC}$. Moreover, $\delta(\overline{AC}/\overline{BC}) = \delta(\overline{AC}/\bar{C}) - e(\overline{AC} - \bar{C}, \bar{C}, \overline{BC} - \bar{C})$.*

Proof. Since $\overline{AC}$ is calculable over $\bar{C}$, there is in increasing chain $(A_n)_{n<\omega}$ of minimal intrinsic extensions with $A_0 = A\bar{C}$ and $\bigcup_{n<\omega} A_n = \overline{AC}$. Put $A'_n = A_n \cup \overline{BC}$. Since $A_n \subset_{\min} A_{n+1}$, we have for any A' with $A'_n \subseteq A' \subset A'_{n+1}$

$$\begin{aligned} \delta(A'_{n+1}/A') &= \delta(A_{n+1}/A') \leqslant \delta(A_{n+1}/A_n(A_{n+1}\cap A')) \\ &= \delta(A_{n+1}/A_n) - \delta(A_{n+1}\cap A'/A_n) < 0. \end{aligned}$$

Hence, after suppressing the elements A'_{n+1} with $A'_{n+1} = A'_n$ of the sequence, it can be refined to an increasing sequence of minimal intrinsic extensions with first element

$A'_0 = A\overline{BC}$ and union $\bigcup_{n<\omega} A'_n = \overline{AC} \cup \overline{BC}$. Thus $\overline{AC}$ is calculable over $\overline{BC}$. Moreover, since $A_n - \overline{BC} = A_n - \bar{C}$,

$$\begin{aligned}
\delta(\overline{AC}/\overline{BC}) &= \lim_{n\to\omega} \delta(A'_n/\overline{BC}) = \lim_{n\to\omega} \delta(A_n/\overline{BC}) \\
&= \lim_{n\to\omega} [\delta(A_n/\bar{C}) - e(A_n - \bar{C}, \bar{C}, \overline{BC} - \bar{C})] \\
&= \lim_{n\to\infty} \delta(A_n/\bar{C}) - \lim_{n\to\infty} e(A_n - \bar{C}, \bar{C}, \overline{BC} - \bar{C}) \\
&= \delta(\overline{AC}/\bar{C}) - e(\overline{AC} - \bar{C}, \bar{C}, \overline{BC} - \bar{C}). \qquad \square
\end{aligned}$$

Now $\overline{AC} \cup \overline{BC}$ is almost closed by Lemma 3.6, since $\overline{AC}$ is calculable over $\overline{BC}$. As $\overline{ABC} = \overline{\overline{AC} \cup \overline{BC}}$, it in turn is calculable over $\overline{AC} \cup \overline{BC}$ by Lemma 3.9. Now by Lemmas 3.8 and 3.13

$$\begin{aligned}
d(A/BC) = \delta(\overline{ABC}/\overline{BC}) &= \delta(\overline{AC} \cup \overline{BC}/\overline{BC}) + \delta(\overline{ABC}/\overline{AC} \cup \overline{BC}) \\
&= \delta(\overline{AC}/\overline{BC}) + \delta(\overline{ABC}/\overline{AC} \cup \overline{BC}) \\
&= \delta(\overline{AC}/\bar{C}) - e(\overline{AC} - \bar{C}, \bar{C}, \overline{BC} - \bar{C}) + \delta(\overline{ABC}/\overline{AC} \cup \overline{BC}) \\
&= d(A/C) - e(\overline{AC} - \bar{C}, \bar{C}, \overline{BC} - \bar{C}) + \delta(\overline{ABC}/\overline{AC} \cup \overline{BC}).
\end{aligned}$$

As $e(\overline{AC} - \bar{C}, \bar{C}, \overline{BC} - \bar{C}) \geqslant 0$ and $\delta(\overline{ABC}/\overline{AC} \cup \overline{BC}) \leqslant 0$,

$$\begin{aligned}
d(A/BC) = d(A/C) \ \text{iff}\ & e(\overline{AC} - \bar{C}, \bar{C}, \overline{BC} - \bar{C}) = 0 = \delta(\overline{ABC}/\overline{AC} \cup \overline{BC}) \\
\text{iff}\ & \overline{AC} \cup \overline{BC} = \overline{AC} \otimes_{\bar{C}} \overline{BC} \ \text{and}\ \overline{AC} \cup \overline{BC} = \overline{ABC} \leqslant M.
\end{aligned}$$

$\square$

Corollary 3.15. *Let $A, B, C \subseteq M$. The following are equivalent*:

(1) $A \downarrow^d_C B$;
(2) $A_0 \downarrow^d_C B_0$ *for all* $A_0 \subseteq_\omega A$ *and* $B_0 \subseteq_\omega B$;
(3) $\overline{A_0C} \cup \overline{B_0C} = \overline{A_0C} \otimes_{\bar{C}} \overline{B_0C} \leqslant M$ *for all* $A_0 \subseteq_\omega A$ *and* $B_0 \subseteq_\omega B$;
(4) $\overline{AC} \cup \overline{BC} = \overline{AC} \otimes_{\bar{C}} \overline{BC} \leqslant M$.

Proof. By the definition and Theorem 3.14, the first three conditions are equivalent. So we have to show that (3) is equivalent to (4). Clearly, $\overline{AC} \cup \overline{BC} = \overline{AC} \otimes_{\bar{C}} \overline{BC}$ iff $\overline{A_0C} \cup \overline{B_0C} = \overline{A_0C} \otimes_{\bar{C}} \overline{B_0C}$ for all $A_0 \subseteq_\omega A$ and $B_0 \subseteq_\omega B$.

Suppose for a contradiction that $\overline{AC} \cup \overline{BC} = \overline{AC} \otimes_{\bar{C}} \overline{BC} \leqslant M$, but there are finite $A_0 \subseteq A$ and $B_0 \subseteq B$ with $\overline{A_0C} \cup \overline{B_0C} \nleqslant M$. Then there exists $F \subseteq_\omega A_0B_0C$ with $\delta(F/\overline{A_0C} \cup \overline{B_0C}) < 0$. Since $\overline{AC} \cup \overline{BC} \leqslant M$, we get $F \subseteq \overline{AC} \cup \overline{BC}$.

Put $F_A = F \cap \overline{AC}$ and $F_B = F \cap \overline{BC}$. Since $\overline{AC} \cup \overline{BC} = \overline{AC} \otimes_{\bar{C}} \overline{BC}$, we have

$$\begin{aligned}
\delta(F/\overline{A_0C} \cup \overline{B_0C}) &= \delta(F_A/\overline{A_0C} \cup \overline{B_0C} \cup F_B) + \delta(F_B/\overline{A_0C} \cup \overline{B_0C}) \\
&= \delta(F_A/\overline{A_0C}) + \delta(F_B/\overline{B_0C}) \geqslant 0,
\end{aligned}$$

a contradiction.

The converse follows from the fact that closure is finitary. $\square$

Corollary 3.16. *$A \downarrow^d_C B$ iff $B \downarrow^d_C A$ iff $A \downarrow^d_{\bar{C}} B$ iff $(A - \bar{C}) \downarrow^d_{\bar{C}} (B - \bar{C})$. Moreover, if $C \subseteq A_0 \leqslant A$ and $C \subseteq B_0 \leqslant B$, then $\overline{AC} \cup \overline{BC} = \overline{AC} \otimes_{\bar{C}} \overline{BC} \leqslant M$ implies $\overline{A_0C} \cup \overline{B_0C} = \overline{A_0C} \otimes_{\bar{C}} \overline{B_0C} \leqslant M$.*

Proof. This is immediate from Corollary 3.15. □

Remark 3.17. It follows that the general definition of d-independence agrees with the definition of d-independence for finite A.

Corollary 3.18. *Let $A, B \subseteq M$. Then there exists a unique minimal closed $C \subseteq \bar{B}$ such that $A \downarrow^d_C B$; moreover $|C| \leqslant |A| + \aleph_0$. In particular, if A is countable, then C is also countable.*

Proof. Suppose first that A is finite. Consider a chain of minimal intrinsic extensions $(A_n)_{n<\omega}$ with $A_0 = A \cup \bar{B}$ and $\bigcup_{n<\omega} A_n = \overline{AB}$. Since $\bar{B}$ is almost closed, $e(A_n - \bar{B}, \bar{B})$ is bounded, and the set $C_n \subseteq \bar{B}$ of all elements in $\bar{B}$ which are in some relation with $A_n - \bar{B}$ is countable. Put $C = \overline{A \bigcup_{n<\omega} C_n} \cap \bar{B}$. By Theorem 3.14 we have $A \downarrow^d_C B$; clearly this C is minimal possible and countable.

Let A be infinite. For any $D \subset_\omega A$ let $C_D \subseteq \bar{B}$ be minimal possible closed with $D \downarrow^d_{C_D} B$, and put $C = \bigcup_{D \subset_\omega A} C_D$. Then C is closed, $A \downarrow^d_C B$, and C is minimal possible. Clearly, $|C| \leqslant |A| + \aleph_0$. □

Now we shall show, for any L-completion T of T_0 having amalgamation over closed sets, the equivalence of non-forking with Φ-independence, as well as stationarity of any type over an algebraically closed set in the real sort. In the finite-closure case, this was done by Wagner [17,19].

Remark 3.19. Let T be an L-completion of T_0. Then T has amalgamation over closed sets iff for any model N of T and any $A, A' \leqslant N$ such that $A \simeq A'$ we have that $\mathrm{tp}_N(A) = \mathrm{tp}_N(A')$.

Proof. (only if): Suppose that N is a model of $T, A_1, A_2 \leqslant N$ and $A_1 \simeq A_2$. Then there exist a model N' of T and elementary embeddings $g_i : N \to N'$ with $g_1(A_1) = g_2(A_2)$ by amalgamation over closed sets. Therefore, $\mathrm{tp}_N(A_1) = \mathrm{tp}_{N'}(g_1(A_1)) = \mathrm{tp}_{N'}(g_2(A_2)) = \mathrm{tp}_N(A_2)$.

(if): Let N_1, N_2 be models of T and let A_i be such that $A_i \leqslant N_i$ for $i = 1, 2$, with $A_1 \simeq A_2$. Take a model N of T such that $N_1, N_2 \preccurlyeq N$. Note that $A_i \leqslant N$ for $i = 1, 2$. Thus $\mathrm{tp}_N(A_1) = \mathrm{tp}_N(A_2)$. We may assume that N is sufficiently saturated, so there exists an elementary embedding $f : N_1 \to N$ mapping A_1 to A_2. □

The following was shown in [4, Theorem 3.34] for theories of generic structures in a finite language with amalgamation over closed sets, but its proof goes through for any L-completion T of T_0 with amalgamation over closed sets.

Remark 3.20. If T is an L-completion of T_0 with amalgamation over closed sets, then T is κ-stable for all cardinals κ with $\kappa = \kappa^{\aleph_0} \geqslant |L|$.

Proof. Let N be a model of T of cardinality κ, where $\kappa^{\aleph_0} = \kappa \geqslant |L|$, and M an elementary extension of N realizing all 1-types over N. By Corollary 3.18, for any element $a \in M$ there exists a countable closed subset $N_a \subseteq N$ such that $a \downarrow^d_{N_a} N$, whence $\overline{aN_a} \cup N = \overline{aN_a} \otimes_{N_a} N \leqslant M$. If $b \in M$ with $N_b = N_a$ and $\overline{bN_a} \simeq \overline{aN_a}$, then $\overline{bN_a} \cup N = \overline{bN_a} \otimes_{N_a} N \leqslant M$, so $\overline{bN_a} \cup N \simeq \overline{aN_a} \cup N$ are two isomorphic closed subsets of M. By Remark 3.19 they have the same type in M, whence $\mathrm{tp}(a/N) = \mathrm{tp}(b/N)$.

There are at most $\kappa^{\aleph_0} = \kappa$ choices for N_a. On the other hand, since on any finite tuple only countably many relations may hold, and as the closure of a countable set is countable, there are at most $(|L|^{\aleph_0})^{\aleph_0} \leqslant \kappa$ choices for the isomorphism type of $\overline{aN_a}$. Therefore $|S_1(N)| \leqslant \kappa$. $\square$

Proposition 3.21. *Let T be an L-completion of T_0, and $\mathscr{M}$ a big model of T.*

(1) *Suppose that T is stable. Let $C \leqslant \mathscr{M}$, let A be an algebraically closed subset of C, and $b \subset_\omega \mathscr{M}$. Then $b \downarrow_A C$ implies $b \downarrow^\Phi_A C$.*
(2) *Suppose that T has amalgamation over closed sets. For any $b \subset_\omega \mathscr{M}$ and algebraically closed sets $A \subset C \subset \mathscr{M}$, we have $b \downarrow_A C$ iff $b \downarrow^\Phi_A C$. Moreover, any type over an algebraically closed set in the real sort is stationary.*

Proof. (1) First we assume that C is an $|A|^+$-saturated model of T. Put $B = \mathrm{acl}(bA)$. By Corollary 3.18 there exists a closed subset A_0 of C of cardinality at most $|B| + \aleph_0$ with $B \downarrow^d_{A_0} C$; clearly $A \subseteq A_0$. Hence $\overline{A_0B} \cup C = \overline{A_0B} \otimes_{A_0} C \leqslant \mathscr{M}$ by Corollary 3.15.

By $|A|^+$-saturation of C, there exists a realization $A_0' \subseteq C$ of $\mathrm{stp}(A_0/A)$ with $A_0' \downarrow_A A_0$. Since $B \downarrow_A C$ we have $B \downarrow_A A_0 A_0'$, whence $A_0 \downarrow_B A_0'$; moreover $\mathrm{tp}(A_0'/B) = \mathrm{tp}(A_0/B)$ by stationarity of strong types, whence $\mathrm{tp}(BA_0) = \mathrm{tp}(BA_0')$. Put $A_0'' = \overline{A_0'B} \cap C$. Then $\overline{A_0''B} = \overline{A_0'B}$, and for any $B_0 \subset_\omega B$ we have

$$d(B_0/C) = d(B_0/A_0) = d(B_0/A_0') \geqslant d(B_0/A_0'') \geqslant d(B_0/C),$$

whence $B \downarrow^d_{A_0''} C$ and $\overline{A_0'B} \cup C = \overline{A_0'B} \otimes_{A_0''} C \leqslant \mathscr{M}$ by Corollary 3.15. Thus

$$\overline{BC} = \overline{A_0B} \otimes_{A_0} C = \overline{A_0'B} \otimes_{A_0''} C \leqslant \mathscr{M}.$$

Since $A_0 \downarrow_B A_0'$ we get

$$B \subseteq \overline{A_0B} \cap \overline{A_0'B} \subseteq \mathrm{acl}(A_0B) \cap \mathrm{acl}(A_0'B) \subseteq \mathrm{acl}(B) = B;$$

as $B \downarrow_A C$ we have $B \cap C = \mathrm{acl}(B) \cap \mathrm{acl}(C) = \mathrm{acl}(A) = A$, whence

$$A \subseteq A_0 \cap A_0'' \subseteq \overline{A_0B} \cap \overline{A_0'B} \cap C = B \cap C = A.$$

Now if $B_iC = B_i \otimes_{A_i} C$ and $B_i \cap C = A_i$ for $i = 1,2$, then intersecting the two sets yields $(B_1 \cap B_2)C = (B_1 \cap B_2) \otimes_{A_1 \cap A_2} C$. Therefore

$$\overline{BC} = (\overline{A_0 B} \cap \overline{A_0' B}) \otimes_{A_0 \cap A_0''} C = B \otimes_A C = BC,$$

and $b \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\Phi}_{A} C$ as desired.

For general C, consider a sufficiently saturated model $N \supseteq C$ with $N \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_C b$. Then $b \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_A N$, whence $\mathrm{acl}(bA) \cup N = \mathrm{acl}(bA) \otimes_A N \leqslant \mathscr{M}$, and $\mathrm{acl}(bA) \cup \mathrm{acl}(AC) = \mathrm{acl}(bA) \otimes_A \mathrm{acl}(AC) \leqslant \mathscr{M}$ by Corollary 3.16, that is $b \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\Phi}_{A} C$.

(2) T is stable by Remark 3.20. Hence non-forking implies Φ-independence. Conversely, suppose $b \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\Phi}_{A} C$. Choose b' realizing $\mathrm{tp}(b/A)$ with $b' \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_A C$. Then $b' \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\Phi}_{A} C$, and $\mathrm{acl}(b'A) \cup \mathrm{acl}(AC) = \mathrm{acl}(b'A) \otimes_A \mathrm{acl}(AC) \leqslant \mathscr{M}$. Therefore $\overline{bAC} \simeq \overline{b'AC}$, and $\mathrm{tp}(bAC) = \mathrm{tp}(b'AC)$ by Remark 3.19. Thus $b \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_A C$.

Finally, we have just seen that $\mathrm{tp}(b/A)$ and $b \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_A C$ together imply $\mathrm{tp}(b/AC)$. But this means that types over an algebraically closed set in the real sort are stationary. □

Corollary 3.22. *Let T be an L-completion of T_0 having amalgamation over closed sets, and $\mathscr{M}$ a big model of T. Let A, B be algebraically closed subsets of $\mathscr{M}$. Then the following are equivalent.*

(1) $A \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{A \cap B} B$,
(2) $A \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\Phi}_{A \cap B} B$,
(3) $A \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{d}_{A \cap B} B$.

Proof. By Corollary 3.15 and Theorem 3.21. □

4. Weak elimination of imaginaries and CM-triviality

In this section, let T be an L-completion of T_0 having amalgamation over closed sets, and $\mathscr{M}$ a big model of T unless stated otherwise.

Wagner suggested a much shorter proof for weak elimination of imaginaries than the second author gave in [19].

Lemma 4.1. *Let A, B, B_1, B_2 be algebraically closed. Suppose that $B_i \subseteq B$ and $A \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{B_i} B$ for $i = 1,2$. Then $A \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{B_1 \cap B_2} B$.*

Proof. Put $A_i = \mathrm{acl}(AB_i)$. Then $A_i B = A_i \otimes_{B_i} B \leqslant \mathscr{M}$ by Corollary 3.22, for $i = 1,2$. Intersecting the two sets yields $(A_1 \cap A_2)B = (A_1 \cap A_2) \otimes_{B_1 \cap B_2} B \leqslant \mathscr{M}$. Note that $A_1 \cap A_2$ and $B_1 \cap B_2$ are algebraically closed. So by Corollary 3.22, $A_1 \cap A_2 \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{B_1 \cap B_2} B$; since $A \subseteq A_1 \cap A_2$ we get $A \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{B_1 \cap B_2} B$, as desired. □

Proposition 4.2. *T has weak elimination of imaginaries.*

Proof. Let $E(\bar{x}, \bar{y})$ be a definable equivalence relation over $\emptyset$, and consider $e = \bar{a}_E$, where $\bar{a}_E$ is the E-class of $\bar{a}$. Take $\bar{b}_1, \bar{b}_2$ such that $\bar{a}, \bar{b}_1, \bar{b}_2$ are independent over e, and $\bar{a}_E = (\bar{b}_1)_E = (\bar{b}_2)_E$. As $e \in \mathrm{acl}^{\mathrm{eq}}(\bar{b}_i)$ we have $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{\bar{b}_i} \bar{b}_1 \bar{b}_2$, for $i = 1,2$.

Put $B = \mathrm{acl}(b_1) \cap \mathrm{acl}(b_2)$, where the algebraic closure is taken in the real sort. Then $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_B \bar{b}_1 \bar{b}_2$ by Lemma 4.1. As $\mathrm{tp}(\bar{a}/B)$ is stationary and $e \in \mathrm{dcl}^{\mathrm{eq}}(\bar{a}) \cap \mathrm{dcl}^{\mathrm{eq}}(\bar{b}_1 \bar{b}_2)$, we get $e \in \mathrm{dcl}^{\mathrm{eq}}(B)$. On the other hand, as $\bar{b}_1 \mathop{\smile\!\!\!\!\!\!\mid}_e \bar{b}_2$, we have $B \subseteq \mathrm{acl}(e)$. By compactness we can find a finite tuple $\bar{b} \in B$ with $e \in \mathrm{dcl}^{\mathrm{eq}}(\bar{b})$; clearly $\bar{b} \in \mathrm{acl}^{\mathrm{eq}}(e)$, as desired. □

We now recall the definition of CM-triviality in stable theories.

Definition 4.3. Let T be stable and $\mathscr{M}$ a big model of T. We say that T is *CM-trivial* if for all $a, A \subset B$ in $\mathscr{M}^{\mathrm{eq}}$, $\mathrm{acl}^{\mathrm{eq}}(aA) \cap \mathrm{acl}^{\mathrm{eq}}(B) = \mathrm{acl}^{\mathrm{eq}}(A)$ implies $\mathrm{Cb}(\mathrm{stp}(a/A)) \subseteq \mathrm{acl}^{\mathrm{eq}}(\mathrm{Cb}(\mathrm{stp}(a/B)))$.

There is a well-known equivalent condition.

Remark 4.4. A stable theory T is CM-trivial iff whenever $B_1, B_2, E \subseteq \mathscr{M}^{\mathrm{eq}}$ are algebraically closed, $B_1 \mathop{\smile\!\!\!\!\!\!\mid}_E B_2$, $\mathrm{acl}^{\mathrm{eq}}(B_1 B_2) \cap \mathrm{acl}^{\mathrm{eq}}(B_i E) = B_i$ for $i = 1, 2$ and $B_1 \cap E = B_2 \cap E$, then $B_1 \mathop{\smile\!\!\!\!\!\!\mid}_{B_1 \cap E} B_2$.

Theorem 4.5. *T is CM-trivial.*

Proof. Let B_1, B_2, E be as in Remark 4.4. By weak elimination of imaginaries in T we may replace $\mathrm{acl}^{\mathrm{eq}}$ by acl. Put $\tilde{B}_i = \mathrm{acl}(B_i E)$ for $i = 1, 2$, and $A = B_1 \cap E$. Then $B_1 \mathop{\smile\!\!\!\!\!\!\mid}_E B_2$ implies $\tilde{B}_1 \cap \tilde{B}_2 = E$, and

$$\tilde{B}_1 \tilde{B}_2 = \tilde{B}_1 \otimes_E \tilde{B}_2 \leqslant \mathscr{M} \quad (\dagger)$$

by Corollary 3.22. By hypothesis $\mathrm{acl}(B_1 B_2) \cap \tilde{B}_i = B_i$ for $i = 1, 2$, and

$$\mathrm{acl}(B_1 B_2) \cap E = \mathrm{acl}(B_1 B_2) \cap \tilde{B}_1 \cap \tilde{B}_2 = B_1 \cap B_2 = B_1 \cap E,$$

as $B_1 \cap E = B_2 \cap E$ and $B_1 \cap B_2 \subseteq \mathrm{acl}(E) = E$. Intersecting (†) with $\mathrm{acl}(B_1 B_2)$ yields

$$B_1 B_2 = B_1 \otimes_{B_1 \cap E} B_2 \leqslant \mathscr{M},$$

and $B_1 \mathop{\smile\!\!\!\!\!\!\mid}_{B_1 \cap E} B_2$ by Corollary 3.22 again. □

We recall the definition of the full amalgamation property.

Definition 4.6. A class $(\mathbf{K}, \leqslant)$ has the full amalgamation property if for any $A, B, C \in \mathbf{K}$ with $A \leqslant B$ and $A = B \cap C$ we have $C \leqslant B \otimes_A C \in \mathbf{K}$ (where $B \otimes_A C$ is the structure with universe BC and whose relations are those of B and of C only).

Remark 4.7 (Wagner [17, Lemma 5.2] and Baldwin and Shi, [4, Lemma 4.2]). $(\mathbf{K}_\alpha, \leqslant)$ has the full amalgamation property.

Fact 4.8 (Baldwin and Shi [4, Lemmas 4.4 and 4.8]). *Suppose the language is finite. If $(\mathbf{K}, \leqslant)$ has the full amalgamation property and M is $(\mathbf{K}, \leqslant)$-generic, then* $\mathrm{Th}(M)$ *has amalgamation over closed sets.*

Corollary 4.9. *In a finite language, if* $(\mathbf{K}, \leqslant)$ *has the full amalgamation property and* M *is* $(\mathbf{K}, \leqslant)$*-generic, then* $\mathrm{Th}(M)$ *is stable CM-trivial with weak elimination of imaginaries. This holds in particular for the theory of a* $(\mathbf{K}_\alpha, \leqslant)$*-generic structure.*

Proof. By Theorem 4.5 and Fact 4.8. □

Appendix A

In this appendix we shall repair the proof [4, Theorem 4.8], which contained many typos. In this section we consider only finite languages.

Review of (Baldwin and Shi [4]).

For a set $A \subseteq M$ let $\delta_A(\bar{x}) = \mathrm{qftp}(A)$.

- For $B \subset_{\min} C$ we write $\varphi_{B,C}(\bar{x}, \bar{y}) := \delta_B(\bar{x}) \wedge \neg\, \delta_C(\bar{x}, \bar{y})$. We also put

$$\mathrm{Diag}_{\leqslant}(N) = \{\forall \bar{y} \varphi_{B,C}(\bar{b}, \bar{y}) : B \subset_{\min} C, N \models \delta_B(\bar{b}),\ N \models \forall \bar{y} \neg \delta_C(\bar{b}, \bar{y})\}.$$

Note that if $N \subset M$ and $M \models \mathrm{Diag}_{\leqslant}(N)$, then $N \leqslant M$.

- A structure $N \in \overline{\mathbf{K}}$ is *full* if for all $A, B, C_1, \ldots, C_n \in \mathbf{K}$ with $A \leqslant B \subset C_i$, $B \not\leqslant C_i$ and $A \leqslant (C_i - B)A$ for $i = 1, \ldots, n$ we have

$$N \models \forall \bar{x} \left\{ \delta_A(\bar{x}) \to \exists \bar{y} [\delta_B(\bar{x}, \bar{y}) \wedge \bigwedge_{i=1}^{n} \neg \exists \bar{z}_i\, \delta_{C_i}(\bar{x}, \bar{y}, \bar{z}_i)] \right\}.$$

We can now prove [4, Theorem 4.8]. In fact, we strengthen it slightly as follows:

Fact 5.1. *Suppose that* $\mathbf{K}$ *has the full amalgamation property, and* $M \in \overline{\mathbf{K}}$ *is full. Then* $\mathrm{Th}(M)$ *has* closed *free amalgamation over closed sets.*

Proof. Let N_1, N_2 be models of $\mathrm{Th}(M)$ and $A \leqslant N_1, N_2$.

Claim. $\mathrm{Th}(N_1, a \in N_1) \cup \mathrm{Diag}_{\leqslant}(N_2) \cup \mathrm{Diag}(N_1 \otimes_A N_2)$ *is consistent.*

Proof of Claim. Consider $\forall \bar{z} \varphi_{B_1,C_1}(\bar{b}_1, \bar{z}) \cdots \forall \bar{z} \varphi_{B_n,C_n}(\bar{b}_n, \bar{z}) \in \mathrm{Diag}_{\leqslant}(N_2)$ and $\Gamma_0(\bar{g}_1, \bar{g}, \bar{g}_2) \in \mathrm{Diag}(N_1 \otimes_A N_2)$, where $\bar{g}_1 \in N_1 - A$, $\bar{g} \in A$ and $\bar{g}_2 \in N_2$. By compactness we need to show

$$N_1 \models \exists y_i \left[\bigwedge_{i=1}^{n} \forall \bar{z}\, \varphi_{B_i,C_i}(\bar{a}_i, \bar{y}_i, \bar{z}) \right] \wedge \exists \bar{y}\, \Gamma_0(\bar{g}_1, \bar{g}, \bar{y}),$$

where $\bar{a}_i = \bar{b}_i \cap A$.

Let $t = \max_i(|C_i - B_i|)$. By [4, Axiom (ii)], given A there is $\varepsilon_t > 0$ such that if $\delta(B/A) < 0$ and $|B - A| \leqslant t$, then $\delta(B/A) < -\varepsilon_t$. Let $G = \{\bar{g}\}, G_1 = \{\bar{g}_1\}$ and $B = \{\bar{b}_i : i = 1, \ldots, n\} \cup \{\bar{g}_2\}$, and put $A_0 = (B \cap A) \cup G \cup G_1$ (which is contained in N_1). Take A_0^*

such that $A_0 \subset A_0^* \subset_\omega N_1$ and $d_{N_1}(A_0) \leqslant \delta(A_0^*) < d_{N_1}(A_0) + \varepsilon_t$, and put $D = A_0^* \otimes_{A_0^* \cap A} B(A_0^* \cap A)$. We have $A \leqslant N_2$ and $A_0^* \cap A = (B(A_0^* \cap A)) \cap A$, whence

$$A_0^* \cap A = (B(A_0^* \cap A)) \cap A \leqslant B(A_0^* \cap A);$$

the full amalgamation property of $(\mathbf{K}, \leqslant)$ yields $A_0^* \leqslant D = A_0^* \otimes_{A_0^* \cap A} B(A_0^* \cap A)$.

Collect all $E \in \mathbf{K}$ (up to isomorphism) such that $(A_0^* \leqslant) D \subset E$, $D \not\leqslant E$, $|E - D| \leqslant t$ and $A_0^* \leqslant (E - D)A_0^*$, and enumerate them as $\{E_i : i \leqslant k\}$. As $A_0^* \subset N_1$ and N_1 is full (note that fullness is preserved under elementary equivalence),

$$\exists \bar{d} \in N_1 [N_1 \models \delta_D(\bar{a}_0^*, \bar{d}) \wedge \bigwedge_i \neg \exists \bar{z}_i \delta_{E_i}(\bar{a}_0^*, \bar{d}, \bar{z}_i)], \qquad (\dagger)$$

where $A_0^* = \{\bar{a}_0^*\}$. Thus, there exists a subset B' of N_1 such that

$$D = A_0^* \otimes_{A_0^* \cap A} B(A_0^* \cap A) \simeq_{A_0^*}^\sigma D' = A_0^* \otimes_{A_0^* \cap A} B'(A_0^* \cap A) = \bar{a}_0^* \bar{d} \subset N_1,$$

where σ is an isomorphism fixing A_0^* pointwise. As $\Gamma_0(\bar{g}_1, \bar{g}, \bar{g}_2)$, $\bar{g}_1, \bar{g} \in A_0$ and $\bar{g}_2 \in D$, there exists $\bar{g}_2' \in D'$ such that $N_1 \models \Gamma_0(\bar{g}_1, \bar{g}, \bar{g}_2')$. Moreover, there is $\bar{b}_i' \in D'$ such that $\bar{b}_i \simeq_{A_0^*}^\sigma \bar{b}_i'$, so $N_1 \models \delta_{B_i}(\bar{b}_i')$.

By way of contradiction suppose that the claim does not hold. Then for some $i \leqslant k$ there exists $\bar{e}_i' \in N_1$ such that $N_1 \models \neg \varphi_{B_i, C_i}(\bar{b}_i', \bar{e}_i')$. As $N_1 \models \delta_{B_i}(\bar{b}_i')$, we see that $N_1 \models \delta_{C_i}(\bar{b}_i', \bar{e}_i)$.

Put $E = \{\bar{a}_0^* \bar{d} \bar{e}_i\}$. As $N_1 \models \delta_{C_i}(\bar{b}_i', \bar{e}_i)$ and $\bar{b}_i' \simeq B_i$, we see that $\bar{b}_i' \bar{e}_i \simeq C_i$. So $\bar{b}_i' \subset_{\min} \bar{b}_i' \bar{e}_i$ and $\delta(\bar{e}_i / \bar{b}_i') < 0$. As $\bar{b}_i' \subset D' \subset E$ and $E - D' = \bar{e}_i$, we get $\delta(\bar{e}_i / D') \leqslant \delta(\bar{e}_i / \bar{b}_i') < 0$. So $A_0^* \leqslant D' \subset E$, $D' \not\leqslant E$, and $|E - D'| \leqslant t$. As $N_1 \models \delta_E(\bar{a}_0^*, \bar{d}, \bar{e}_i)$, we see that $A_0^* \not\leqslant (E - D')A_0^*$ by $(\dagger)$.

Let F be such that $A_0^* \subset F \subseteq (E - D')A_0^*$ and $\delta(F / A_0^*) < 0$. As $|F - A_0^*| \leqslant t$, we see that $\delta(F / A_0^*) < -\varepsilon_t$. Since $A_0^* \subset F \subset_\omega N_1$, we get

$$d_{N_1}(A_0^*) \leqslant \delta(F) = \delta(A_0^*) + \delta(F / A_0^*) < d_{N_1}(A_0^*) + \varepsilon_t - \varepsilon_t.$$

This contradiction proves the claim. □

Let $N' \models \mathrm{Th}(N_1, a \in N_1) \cup \mathrm{Diag}_\leqslant(N_2) \cup \mathrm{Diag}(N_1 \otimes_A N_2)$.

Claim. $\mathrm{Th}(N', a \in N') \cup \mathrm{Diag}_\leqslant(N_1 \otimes_A N_2)$ *is consistent.*

Proof of Claim. Let $\forall \bar{z} \varphi_{B_1, C_1}(\bar{b}_1, \bar{z}) \cdots \forall \bar{z} \varphi_{B_n, C_n}(\bar{b}_n, \bar{z}) \in \mathrm{Diag}_\leqslant(N_1 \otimes_A N_2)$, where $\bar{b}_i \in N_1 \otimes_A N_2$. By compactness we need to show

$$N' \models \bigwedge_{i=1}^n \forall \bar{z} \varphi_{B_i, C_i}(\bar{b}_i, \bar{z}).$$

Let $t = \max_i(|C_i - B_i|)$, and take $\varepsilon_t > 0$ as in the proof of the first claim. Let $B = \{\bar{b}_i : i = 1, \ldots, n\} \cap N_1$, and put $A_0 = \{\bar{b}_i : i = 1, \ldots, n\} \cap N_2 (\subset N_2)$. Then take A_0^* such that $A_0 \subset A_0^* \subset_\omega N_2$ and $d_{N_2}(A_0) \leqslant \delta(A_0^*) < d_{N_2}(A_0) + \varepsilon_t$. Put $D = A_0^* \otimes_{A_0^* \cap A} B(A_0^* \cap A)$. As $A \leqslant N_1$

and $A_0^* \cap A = (B(A_0^* \cap A)) \cap A$, we have $A_0^* \cap A = (B(A_0^* \cap A)) \cap A \leqslant B(A_0^* \cap A)$. Hence the full amalgamation property yields

$$A_0^* \leqslant D = A_0^* \otimes_{A_0^* \cap A} B(A_0^* \cap A) \qquad (\ddagger).$$

Collect all $E \in \mathbf{K}$ (up to isomorphism) such that $(A_0^* \leqslant) D \subset E$, $D \not\leqslant E$, $|E - D| \leqslant t$ and $A_0^* \leqslant (E - D)A_0^*$, and enumerate them as $\{E_i : i \leqslant k\}$. As $A_0^* \subset N'$ and N' is full,

$$\exists \bar{d} \in N' \left[N' \models \delta_D(\bar{a}_0^*, \bar{d}) \wedge \bigwedge_i \neg \exists \bar{z}_i \delta_{E_i}(\bar{a}_0^*, \bar{d}, \bar{z}_i) \right],$$

where $A_0^* = \{\bar{a}_0^*\}$. Thus there exists a subset B' of N' such that

$$D = A_0^* \otimes_{A_0^* \cap A} B(A_0^* \cap A) \simeq^{\sigma}_{A_0^*} D' = A_0^* \otimes_{A_0^* \cap A} B'(A_0^* \cap A) = \bar{a}_0^* \bar{d} \subset N',$$

where σ is an isomorphism fixing A_0^* pointwise. Moreover, there is $\bar{b}_i' \in D'$ with $\bar{b}_i \simeq^{\sigma}_{A_0^*} \bar{b}_i'$, so $N' \models \delta_{B_i}(\bar{b}_i')$.

By way of contradiction suppose that the claim does not hold. Then for some $i \leqslant k$ there exists $\bar{e}_i' \in N'$ such that $N' \models \neg \varphi_{B_i, C_i}(\bar{b}_i', \bar{e}_i')$. As $N' \models \delta_{B_i}(\bar{b}_i')$, we see that $N_1 \models \delta_{C_i}(\bar{b}_i', \bar{e}_i)$. Put $E = \{\bar{a}_0^* \bar{d} \bar{e}_i\}$. As $N' \models \delta_{C_i}(\bar{b}_i', \bar{e}_i)$ and $\bar{b}_i' \simeq B_i$, we have $\bar{b}_i' \bar{e}_i \simeq C_i$. So $\bar{b}_i' \subset_{\min} \bar{b}_i' \bar{e}_i$ and $\delta(\bar{e}_i / \bar{b}_i') < 0$. As $\bar{b}_i' \subset D' \subset E$ and $E - D' = \bar{e}_i$, we get $\delta(\bar{e}_i / D') \leqslant \delta(\bar{e}_i / \bar{b}_i') < 0$. So $A_0^* \leqslant D' \subset E$, $D' \not\leqslant E$, and $|E - D'| \leqslant t$. As $N' \models \delta_E(\bar{a}_0^*, \bar{d}, \bar{e}_i)$, we get $A_0^* \not\leqslant (E - D')A_0^*$ by ($\ddagger$).

Let F be such that $A_0^* \subset F \subseteq (E - D')A_0^*$ and $\delta(F / A_0^*) < 0$. As $|F - A_0^*| \leqslant t$, we see that $\delta(F / A_0^*) < -\varepsilon_t$. As $A_0^* \subset F \subset_{\omega} N'$ and $N_2 \leqslant N'$, and since $d_{N_2} = d_{N'}$ inside N_2, we have by [4, Lemma 3.15]

$$d_{N_2}(A_0^*) = d_{N'}(A_0^*) \leqslant \delta(F) = \delta(A_0^*) + \delta(F / A_0^*) < d_{N_2}(A_0^*) + \varepsilon_t - \varepsilon_t = d_{N_2}(A_0^*);$$

this contradiction proves the claim. □

Similarly, we can show that $\mathrm{Th}(N_2, a \in N_2) \cup \mathrm{Diag}_{\leqslant}(N_1) \cup \mathrm{Diag}_{\leqslant}(N_1 \otimes_A N_2)$ is consistent. Finally, we construct elementary chains as follows:

$$N_1^{i+1} \models \mathrm{Th}(N_1^i, a \in N_1^i) \cup \mathrm{Diag}_{\leqslant}(N_2^i) \cup \mathrm{Diag}_{\leqslant}(N_1^i \otimes_A N_2^i),$$

$$N_2^i \models \mathrm{Th}(N_2^{i-1}, a \in N_2^{i-1}) \cup \mathrm{Diag}_{\leqslant}(N_1^i) \cup \mathrm{Diag}_{\leqslant}(N_1^i \otimes_A N_2^{i-1}).$$

As

$$N_1 = N_1^0 \preccurlyeq N_1^1 \preccurlyeq N_1^2 \preccurlyeq \cdots \preccurlyeq N_1^i \preccurlyeq \cdots,$$

$$N_2 = N_2^0 \preccurlyeq N_2^1 \preccurlyeq N_2^2 \preccurlyeq \cdots \preccurlyeq N_2^i \preccurlyeq \cdots,$$

and

$$N_1, N_2 \subset N_1^i \leqslant N_2^i \leqslant N_1^{i+1} \leqslant \cdots \quad (\text{for } i \geqslant 1),$$

we see that

$$\bigcup_{i<\omega} N_1^i = \bigcup_{i<\omega} N_2^i \models \mathrm{Th}(N_1, a \in N_1) \cup \mathrm{Th}(N_2, a \in N_2) \cup \mathrm{Diag}_{\leqslant}(N_1 \otimes_A N_2). \qquad \square$$

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